*Reply by R. L. Kyhl*

I have no disagreement with the comments of L. J. Kaplan and D. J. R. Stock. I was thinking of plotting the two parts of the chart from different origins. My chief interest was in the type of graphical display chosen.

While the issue was at the press, a similar use of a double chart was published elsewhere.[10]

[10] R. M. Steere, "Novel applications of the Smith chart," *Microwave J.*, vol. 3, pp. 97–100; March, 1960.

## Scattering Matrix for an N-Port Power-Divider Junction*

### Introduction

During the course of an investigation of a data-processing technique yielding effectively reduced sidelobes and beamwidth for a microwave radar antenna, the need arose for multiport power dividers. In order to avoid an undesirable decrease in the signal-to-noise ratio, it was necessary that these dividers waste no power. Consequently, a scattering matrix was sought which would have the obvious requirement that there be no wave reflected in the input port and which would allow the power to be divided into arbitrary but fixed relative parts.

It should be noted that there now exist two methods[1] for synthesizing an $n$-port junction at a single frequency directly from the normalized scattering matrix, without use of the associated impedance matrix.

### The Scattering Matrix

A reciprocal, lossless junction can be represented by a symmetric, unitary scattering matrix $\mathbf{S}'$. It is sufficient to consider a real matrix first without losing generality, because the matrix $\mathbf{S}'$ for the general case (including phase shifts) can be derived from the real case by a simple transformation.[2]

The purpose of this paper is to find the scattering matrix for an $n$-port junction, such that when a wave is fed into, say, port one, there will be no reflected wave in that port, and such that the amplitude of the wave transmitted to port $k$ is equal to a given $x_k$. Analytically expressed, this requires that

$$x_k = \sum_i S_{ki}\delta_{1i} = S_{k1}, \tag{1}$$

where the $x_k$ are subject to the restriction

$$x_1 = 0 \quad \text{and} \quad \sum_{k=2}^{n} x_k^2 = 1. \tag{2}$$

Symmetry requires that

$$S_{ij} = S_{ji}, \tag{3}$$

and unitarity, which reduces to orthogonality for the real case considered here, requires that

$$\sum_{k=1}^{n} (\tilde{S})_{ik} S_{kj} = \sum_{k=1}^{n} S_{ik} S_{kj} = \delta_{ij} \tag{4}$$

[where use has been made of (3)].

the matrix $\mathbf{S}$ satisfying (1), (3), and (4) may be obtained as follows:

$$\left.\begin{array}{ll} \text{A. } S_{11} = 0, & \\ \text{B. } S_{kk} = x_k^2 - 1 & \text{for } k > 1, \\ \text{C. } S_{ij} = S_{ji} = x_i x_j & \text{for } i \neq j, i > 1, j > 1, \\ \text{D. } S_{1i} = S_{i1} = x_i & \text{for } i > 1. \end{array}\right\} \tag{5}$$

To prove that (5) is the required solution, it is only necessary to verify that it satisfies (1), (3), and (4). From $A$ and $D$, it is evident that (1) is satisfied, and from $C$ and $D$ it is evident that (3) is satisfied. To verify that (4) is also satisfied, it is necessary to consider separately various possible values of $i$ and $j$, because of the special nature of the various $S_{ij}$.

1) $i=j=1$.

Using (2) and the rules given in (5),

$$\sum_{k=1}^{n} S_{1k} S_{k1} = \sum_{k=2}^{n} x_k^2 = 1,$$

which satisfies (4).

2) $i=j\neq 1$.

Proceeding as above,

$$\sum_{k=1}^{n} S_{ik} S_{ki} = S_{i1}^2 + S_{ii}^2 + \sum_{k\neq 1,i} S_{ik} S_{ki} = x_i^2 + (x_i^2 - 1)^2 + x_i^2 \sum_{k\neq i} x_k^2.$$

According to (2), the above sum on $k$ is $(1-x_i^2)$ and hence the right side reduces to one as required.

3) $i\neq j$.

$$\sum_{k=1}^{n} S_{ik} S_{kj} = S_{i1} S_{1j} + S_{ii} S_{ij} + S_{ij} S_{jj} + \sum_{k\neq 1,i,j} S_{ik} S_{kj}. \tag{6}$$

It is necessary to consider separately the case $i=1$ (or $j=1$) and $i\neq 1\neq j$.

a) $i=1$.

For this case, the above expression rereduces to

$$\sum_{k=1}^{n} S_{1k} S_{kj} = x_j(x_j^2 - 1) + x_j \sum_{k\neq j} x_k^2 = 0.$$

The case $j=1$ is essentially the same as that above and therefore (4) is satisfied in both of these cases.

b) $i\neq 1\neq j$.

For this case, (6) becomes

$$\sum_{k=1}^{n} S_{ik} S_{kj} = x_i x_j + x_i x_j (x_i^2 + x_j^2 - 2) + x_i x_j \sum_{k\neq i,j} x_k^2.$$

When use is made of (2), the above equation reduces to zero as required.

### Conclusion

It has been shown that it is theoretically possible to provide a junction which will divide an input wave into many output waves of arbitrary but fixed relative amplitudes. The arbitrariness of the power division means that one may choose any ratios for the output waves and have no reflected wave in the input port. Once such a divider is constructed, however, the ratio of the output waves is fixed.

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* Received by the PGMTT, July 15, 1960.

[1] D. C. Youla, "Direct single frequency synthesis from a prescribed scattering matrix," IRE Trans. on Circuit Theory, vol. CT-6, pp. 340–344; December, 1959.

[2] "Reference Data for Radio Engineers," American Book—Stratford Press, Inc., New York, N. Y., 4th ed.; 1956.

## Lossy Resonant Slot Coupling*

The paper by Allen and Kino[1] suggests a novel method of combating troublesome cutoff oscillations in periodic slow-wave structures. The idea is to couple loss periodically into the system through slots which are resonant at the center of the (narrow) oscillation range. The high $Q$ of the slots will effectively decouple the loss in the operating range of the pass band.

To develop this idea, we start with Allen and Kino's (13) for the voltage $\phi(x)$ along the slot in terms of the tangential $\hat{H}$ field along the slot. With respect to their Fig. 2 coordinates, shown in Fig. 1 below, we can

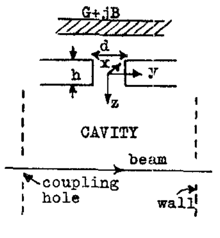


Fig. 1—A lossy slot in a cavity wall. Voltage $\phi(x)$ exists across the slot gap, $d$.

write the transmission line equation for the slot voltage as

$$(\partial^2/\partial x^2 + k^2)\phi(x) = -j\omega L_0 [(I_c \hat{H}_{cx})_+ - \hat{H}_{x-}] \tag{1}$$

where the tangential magnetic field is $(I_c \hat{H}_{cx})_+$ on the $+z$- or cavity side of the slot and is $\hat{H}_{x-}$ on the $-z$-side. Eq. (1) can be derived rigorously for a TEM slot mode. The caret denotes a total field, and we have split the cavity field into an amplitude $I_c$ and a vector field pattern $\hat{H}_c$ for equivalent circuit purposes; this notation differs from that of Allen and Kino. $L_0$ is the slot inductance per unit length in the $x$-direction.

Let us introduce the lossy susceptance $G+jB$ by saying that the average voltage

* Received by the PGMTT, July 20, 1960.

[1] M. A. Allen and G. S. Kino, "On the theory of strongly coupled cavity chains," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 362–372; May, 1960.

across the gap, $\phi(x)/d$, produces $\hat{H}_{x-}$ according to

$$\hat{H}_{x-}d = \phi(x)(G + jB) \qquad (2)$$

so that (1) may be written as

$$(\partial^2/\partial x^2 + \kappa^2)\phi(x) = -j\omega L_0 I_c \bar{H}_{cx}$$
$$\kappa^2 = k^2 - j\omega \frac{L_0}{d}(G + jB)$$
$$\cong k^2 - j\omega(L_0/d)G. \qquad (3)$$

(Note that $L_0/d$ is well-behaved as $d$ approaches zero because $L_0 \cong (\mu d/h)$. It will turn out that we will need very little conductance $G$ for our purposes; hence $\omega(L_0/d)G \ll k^2$, and since $B$ is presumed on the order of $G$, we have neglected $\omega(L_0/d)B$ above.

Eq. (3) is our transmission line equation for slot voltage $\phi(x)$, with a loss term representing decay of a wave traveling along the slot $x$-axis. The equation says that all power flowing into the slot from the cavity is absorbed by $G$. To prove this, merely multiply (3) by $\phi(x)^*$ and integrate from $x = -l/2$ to $+l/2$. By integration of the first term by parts we obtain

power from cavity

$$= 1/2 \operatorname{Re} I_c^* \bar{H}_{cx} \int_{-l/2}^{+l/2} \phi(x)dx$$
$$= 1/2(G/d)\int_{-l/2}^{+l/2} |\phi(x)|^2 dx \qquad (4)$$

= power into $G$, consistent with (2).

We shall use the first expression for power flow rather than the second.

Webster's solution (15) of the reference reduces, for the case of $I_c\bar{H}_{cx}$ constant over the long narrow slot, to

$$\phi(x) = +j\omega L_0 I_c \bar{H}_{cx} \frac{2 \sin \kappa l/2}{\kappa^2 \sin \kappa l} \cdot [\cos \kappa x - \cos \kappa l/2] \qquad (5)$$

which correctly reduces to zero at the ends of the slot, $x = \pm l/2$.

With slot voltage $\phi(x)$ known in terms of $I_c$ from (5), we now want to determine cavity mode amplitude $I_c$ in terms of $\phi(x)$. Suppose the unwanted oscillations occur near the $\pi$-cutoff frequency of the first pass band of a cavity chain with hole coupling from one cavity to the next. In such a "cold" structure of $\pi$-phase shift, the magnetic field will be purely normal and the electric field purely tangential to the cavity coupling holes. We conveniently define the cavity $\pi$-mode patterns $\bar{E}_c$ and $\bar{H}_c$ as

$$\nabla \times \bar{E}_c = k_\pi \bar{H}_c \qquad (6a)$$
$$\nabla \times \bar{H}_c = k_\pi \bar{E}_c \qquad (6b)$$
$$\int_v \bar{E}_c^2 dv = \int_v \bar{H}_c^2 dv = \tau,$$

cavity volume (normalization) (6c)

with

$\bar{E}_c \cdot \bar{n} = 0$ on coupling holes,

$\bar{E}_c \times \bar{n} = 0$ on the "lossy" slot surface and metal walls.

In a frequency range near the $\pi$-cutoff, we expand the fields of Maxwell's equations, but not their curls, in the one-$\pi$ mode, as

$$\nabla \times \hat{E} = -j\omega\mu(I_c\bar{H}_c), \qquad (7a)$$
$$\nabla \times \hat{H} = \hat{J} + j\omega\epsilon(V_c\bar{E}_c). \qquad (7b)$$

The interaction of the beam current density $\hat{J}$ with the circuit near the cutoff causes the unwanted oscillations. Eqs. (7a) and (7b) neglect the irrotational field, which is weak compared to the one strongly-excited solenoidal $\pi$-mode.

To evaluate $V_c$ and $I_c$ of (7) we dot-multiply (7a) by $\bar{H}_c$; (7b) by $\bar{E}_c$; integrate both equations over the cavity volume, and integrate the curl $\hat{E}$ and curl $\hat{H}$ terms by parts, *viz.*,

$$\int_v \bar{H}_c \cdot \nabla \times \hat{E} dv$$
$$= \int_v \hat{E} \cdot \nabla \times \bar{H}_c dv + \int_{\text{slot}} \hat{E} \times \bar{H}_c \cdot \bar{n} ds$$
$$= \int_v V_c \bar{E}_c \cdot (k_\pi \bar{E}_c) dv + \int_{\text{slot}} \hat{E} \times \bar{H}_c \cdot \bar{n} ds. \qquad (8)$$

We obtain the equations for amplitudes $V_c$ and $I_c$ as

$$k_\pi V_c = -(1/\tau)\int_{\text{slot}} \bar{i}_y \phi(x) \times \bar{i}_x H_{cx} \cdot (-\bar{i}_z) ds - j\omega\mu I_c \qquad (9a)$$
$$k_\pi I_c = j\omega\epsilon V_c + k_\pi \Delta I \qquad (9b)$$

with

$$k_\pi \Delta I \triangleq (1/k_\pi \tau)\left[\int_v \hat{J} \cdot \bar{E}_c dv + \int_{\text{coupling holes}} \bar{E}_c \times \hat{H} \cdot \bar{n} ds\right].$$

Note that the last integral in $k_\pi \Delta I$ represents the cavity hole coupling to its neighbors by means of tangential $\hat{H}$ on the hole surfaces. Thus we do not consider resonant slot coupling between cavities, effected by tangential $\hat{E}$ on the slot surfaces. Next, we will compare the power flow through the coupling holes with that dissipated across the slot in such a way that $k_\pi \Delta I$ will be irrelevant in (9). But first we derive the equivalent circuit.

If we divide both parts of (9) by $k_\pi^2$ and define equivalent circuit quantities as

$$V_c' \triangleq V_c/k_\pi \text{ (volt)} \qquad L_c \triangleq \mu/k_\pi$$
$$I_c' \triangleq I_c/k_\pi \text{ (ampere)} \qquad C_c \triangleq \epsilon/k_\pi \qquad (10a)$$

$Z_s$, slot impedance

$$= +j\omega \frac{L_0}{k_\pi} \frac{H_{cx}^2}{\tau} \frac{2 \sin \kappa l/2}{\kappa^3 \sin \kappa l} \cdot \left[2 \sin \frac{\kappa l}{2} - \kappa l \cos \frac{\kappa l}{2}\right] \qquad (10b)$$

$$\kappa \cong k - j\omega \frac{L_0}{2k}\frac{G}{d}, \qquad (10c)$$

from (3) for small loss, then (9) and (10) give us

$$V_c' = -Z_s I_c' - j\omega L_c I_c' \qquad (11a)$$
$$I_c' = j\omega C_c V_c' + \Delta I/k_\pi, \qquad (11b)$$

which has the equivalent circuit representation shown in Fig. 2 below. It may be verified that the time average stored energies and the power flow into Re $(Z_s)$ of the circuit are equal to the actual ones times the dimensionless factor $(k_\pi^3\tau)^{-1}$. Closer examination would reveal that the power flow $\frac{1}{2}$ Re $V_c'(\Delta I/k_\pi)^*$ represents flow into the beam and across the coupling holes by the same factor of proportionality.

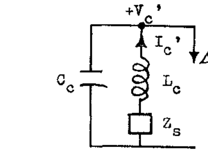


Fig. 2—Equivalent circuit for the cavity excited by the slot. $Z_s$ is the slot impedance. The effect of the beam current and coupling to adjacent cavities is represented by $\Delta I/k_\pi$.

Now we note the fact that the electromagnetic power flowing into the cavity across a coupling hole (through which the beam may be flowing) is proportional to the product of amplitude of electric field $V_c$, and $I_c^*$, which is proportional to $V_c^*$ in (11a). Therefore the electromagnetic power flow in the chain is proportional to $|V_c'|^2$ in the equivalent circuit. Since the power dissipated into Re $(Z_s)$ is also proportional to $|V_c'|^2$, we see that the significant parameter for comparison of power input to a cavity with power dissipated is

$$P_d'/|V_c'|^2 = \frac{1}{2} \frac{\operatorname{Re} Z_s}{|Z_s + j\omega L_c|^2}, \qquad (12)$$

$P_d'$ being the equivalent circuit power dissipation across the lossy slot. We shall call the ratio of (12) the relative power dissipation.

We now show that as conductance $G$ of the lossy material goes to zero, the relative power dissipation tends to concentrate in a narrow frequency range about the slot resonant frequency. Let us define the loss parameter $\alpha$ as $(\omega L_0 G/2dk)$ from (10c). For small loss, $|Z_s| \ll \omega L_c$ in (12) and Re $(Z_s)$ varies rapidly about $kl = \pi$, the first half-wavelength resonance of the slot, so we write $P_d'/V_c'|^2$ as

$$P_d'/|V_c'|^2 = \frac{1}{2}\frac{1}{(\omega L_c)^2}\frac{\omega L_0}{k_\pi}\frac{H_{cx}^2 l^3}{\tau}\frac{2}{(kl)^3}R_s' \qquad (13a)$$

$$R_s' = \operatorname{Re}\frac{j \sin \kappa l/2}{\sin \kappa l}[2 \sin \kappa l/2 - \kappa l \cos \kappa l/2] \qquad (13b)$$

$$\kappa l = kl - j\alpha l \qquad \alpha = \frac{\omega L_0}{2k}\left(\frac{G}{d}\right).$$

Plots of $R_s'$ as a function of the slot length $kl$, in radians, appear on Fig. 3 for various losses $\alpha l$. Because $|Z_s| \ll \omega L_c$ in the range of $R_s'$ shown, the equivalent circuit current $I_c'$ in Fig. 2 remains constant for fixed $|V_c'|$ but as $\alpha l$ decreases, $G$ decreases and Re $(Z_s)$ increases, hence the relative power dissipation at $\pi$ increases with decreasing $\alpha l$ on Fig. 3. Note that in describing the behavior of these parameters we have not involved the resonant circuit properties of the Fig. 2 circuit;

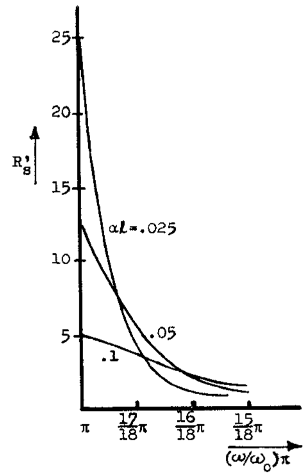


Fig. 3—Relative power dissipation measured by $R_s'$ vs frequency $\omega$, for various values of the loss parameter $\alpha l$.

hence, the Fig. 3 behavior in no way contradicts the behavior of $Q$ with loss and frequency.

A properly designed lossy slot will not have the relative power dissipation so large at the slot resonant frequency $\omega_0$ that sufficient power will dissipate within the pass band to interfere with operation there. Nor should it be so small that sufficient residual power remains to start oscillation. One design procedure is the following:

1) Decide from a calculation of power flow across the coupling holes how much relative power $P_d/|V_c|^2$ is to be dissipated per cavity. Convert this to the equivalent circuit relative dissipation by

$$P_d'/|V_c'|^2 = (P_d/k_\pi^3\tau)(k_\pi^2/|V_c|^2) = (P_d/|V_c|^2)(1/k_\pi\tau). \quad (14)$$

($V_c$ is the amplitude of electric field in volts/meter.)

2) Choose a loss parameter

$$\alpha l = (\omega\mu l/2k)(L_0/\mu)(G/d)$$

from Fig. 3 for the desired power dissipation bandwidth about the half-wavelength frequency $\omega_0$ of the slot. This expression involves two parameters: $(L_0/\mu)$, the slot transmission line inductance/unit length along the long axis divided by $\mu$, and $(G/d)$, the ratio of the lossy surface conductance presented to the slot divided by the gap length in Fig. 1.

3) From the known parameters $k_\pi$; $H_{cx}$ at the slot; $\omega\mu$; $l$, the slot length; and $\tau$, the cavity volume, choose $(L_0/\mu) \cong (d/h)$ from (10b) and (12) for sufficient relative power dissipation at frequency $\omega_0$.

4) Return to the expression for $\alpha l$ in 2) and, with $L_0/\mu$ now known, determine $G/d$.

We shall see next that the relative power dissipation at $\omega_0$ tends to be unduly large unless $L_0/\mu \cong d/h$ of the slot is designed rather small.

Some numerical results now follow. Suppose we have the parameters

$$\text{cavity}\begin{cases} k_\pi = 75 \sim 3.6 \text{ kmc} \\ H_{cx} = 1.1^2 \\ l = 4.15 \times 10^{-2} \text{ meter} \\ \tau = 12.2 \times 10^{-5} \text{ meter}^3 \end{cases}$$

$$\text{beam}\begin{cases} \text{radius } 1/4 \text{ inch} \\ \text{current } 50 \text{ amperes} \\ \text{voltage } 100 \text{ kv.} \end{cases}$$

We estimate from the gain per cavity of the small-signal growing wave, which exists for synchronism of the slow space charge mode with the "cold" circuit near the $\pi$-cutoff, a value of $P_{\text{flow}}/|V_c|^2$ of $6.8\times10^{-7}$. Let us dissipate this amount of relative power through the slot, so that $P_d/|V_c|^2$ has this value. We choose $\alpha l = 0.025$ in Fig. 3 so that the slot dissipation will be effective over 10° or so, corresponding to $\Delta f = 0.2$ kMc. From (14) we get $P_d'/|V_c'|^2 = 7.5\times10^{-5}$. Then, from (13) we find $(L_0/\mu) \cong d/h$ to be about 0.052. This value is impractically small but we can insert a lossy slot into every fifth cavity of the chain and dissipate five times the relative power in that cavity. Then $d/h$ becomes about 0.26. From (13b) for $\alpha$ we get a value of $G/d$ of about $1\times10^{-2}$. For a $\frac{1}{32}$-inch-wide slot, $G = 10^{-5}$ mho per unit surface area.

Despite the critical nature of the relative power dissipation and the loss bandwidth upon $G$, the idea of lossy slot coupling for combating undesirable cutoff oscillations looks promising.


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## Authors' Comment*

The idea suggested by Bevensee for coupling energy out of a periodic system by a means which is effective only over a narrow pass band is one of many possibilities which have been considered in this Laboratory. Another alternative, described by Rynn,[3] for instance, involves the use of another propagating structure which is itself lossy and which is coupled to the main structure. In this case, conditions are arranged for the phase velocities of the two structures to be equal over only the narrow oscillation pass band.

A note of caution should be introduced. It will be realized from our analysis that a long slot is capable of presenting a large impedance over a wide frequency band. Consequently, if additional slots are cut into every fifth cavity of the system, as Bevensee suggests, there is the strong possibility that new stop bands will be introduced in the region of $\pi/5$, $2\pi/5$, $3\pi/5$, and $4\pi/5$ phase shift between cavities, and cutoff oscillations may occur at the corresponding frequencies. The introduction of extra resonant elements in an already complicated propagating system can sometimes hinder rather than help.


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## Broad-Band Hybrid Junctions*

A coaxial version of the wide-band strip-line magic T described by E. M. T. Jones in the March, 1960 issue of these Transactions was developed jointly at the Mullard Research Laboratories and the Laboratories of the General Electric Company, England, some years ago. It has proved a powerful component in the design of broad-band receiver circuits and has the advantage of small size and geometrical symmetry in addition to wide frequency coverage.

In particular, the circuit has been used in broad-band balanced mixers, balanced modulators, single side-band modulators, limiters and isolating power splitters. In each case, the property of a 180° hybrid, by which isolation is independent of the value of two balanced terminating impedances, gives the device advantages over the more usual broad-band 90° hybrid circuits. Mullard balanced mixers type L361 ($S$-band) and L360 ($X$-band) are examples of the commercial application of the circuit.

Devices have been successfully operated in the frequency range 1.0–11.5 kMc, and for many applications it has proved convenient to divide this range into five overlapping bands.

The simplest embodiment of the hybrid is used, in which one arm contains a shorted coupled line filter section and the other three arms are simple lines for which $\theta = \pi/2$ at midband. In this case, one set of conditions for satisfactory operation is

$$Z_1 = Z_2 = 0.71Z$$

and

$$Z_{oo} = Z_{oe} = 1.33Z,$$

whence

$$\theta = \beta = \frac{\pi}{2}$$

at midband, where

$\theta$ = the electrical length of the three similar arms,
$\beta + \pi$ = the electrical length of the filter section,
$Z$ = the characteristic impedance of the three similar arms,
$Z_{oo}$ = the characteristic impedance of the unbalanced mode,
$Z_{oe}$ = the characteristic impedance of the balanced mode,
$Z_1 = Z_2$ = the hybrid terminating impedances.

Circuits with open-coupled line sections and other systems of compensation have not justified the additional mechanical complexity. Moreover, the basic circuit has the advantage of superior isolation symmetry between the various terminals. If an attempt is made to recover this symmetry by compensating all three remaining arms, instead of one as described, extremely broad-band isolation characteristics can be obtained, but the input match of the device suffers.

Typical theoretical and experimental performance figures for the basic circuit over

[2] $\bar{H}_c$ is easily estimated from the formula for $\bar{H}_c$ of the $TM_{010}$ mode of a closed cylindrical cavity, satisfying (6). It is $i_\phi\sqrt{3.7}J_1(k_1r)$, where $k_1 = 2.4/$(cavity radius).

[3] N. Rynn, "On the periodic coupling of propagating structures," IRE Trans. on Electron Devices, vol. ED-6, pp. 325–329; July, 1959.

* Received by the PGMTT, August 8, 1960.